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On point transformations of generalized nonlinear diffusion equations

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Abstract. This paper classifies all finite point transformations of a general class between generalized diffusion equations of the form $u_t = x^{1-M}[x^{N-1}f(u)u_x]_x$. These transformations may be divided into three cases, depending on the functional form of $f(u)$: (i) f arbitrary, (ii) $f = u^n$ and (iii) $f = e^u$. In particular, these transformations include all the invariant infinitesimal transformations and, in addition, they include a number of point transformations which relate different equations of the above form. Many exact solutions are already known and the transformations which are derived here may be used to obtain new solutions from these.

1. Introduction

We consider the generalized radially symmetric diffusion equations of the form

$$\frac{\partial u}{\partial t} = x^{1-M_1}\frac{\partial}{\partial x}\left[x^{N_1-1}f(u)\frac{\partial u}{\partial x}\right] \tag{1}$$

which are of considerable interest in mathematical physics. In some special cases they have been used to model physical situations in some fields involving diffusion processes [1–3]. There is a continuing interest in finding exact similarity solutions to these equations [4–7].

In [8] it is pointed out that the point transformation

$$t' = t \qquad x' = \frac{2}{2+M_1-N_1}x^{\frac{2+M_1-N_1}{2}} \qquad u' = u \tag{2}$$

where $M_1 - N_1 + 2 \neq 0$, transforms (1) into radially symmetric nonlinear diffusion equations of the form

$$\frac{\partial u'}{\partial t'} = x'^{1-N}\frac{\partial}{\partial x'}\left[x'^{N-1}f(u')\frac{\partial u'}{\partial x'}\right] \tag{3}$$

where $N = 2M_1/(2+M_1-N_1)$. The symmetries and the point transformations, in general of (3) are presented in [8,9], respectively. Nevertheless, consideration of (1) will lead to more point transformations. In particular, examination of the case $M_1 - N_1 + 2 = 0$ leads to some interesting point transformations.

In the next section equation (1) and a general class of point transformations are considered and a set of functional equations is derived, which are then investigated for three exclusive cases depending on the form of $f(u)$. These cases are: (i) $f(u)$ arbitrary, (ii) $f(u) = u^n$ and (iii) $f = e^u$. It can be shown that these are the only forms of $f(u)$ that produce such transformations for (1) (see, for example, [9]). In the final section we present examples where we use known solutions of (1) and point transformations which are derived in section 2, to obtain new solutions.

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2. Point transformations

We consider the non-degenerate point transformation

$$x' = P(x,t) \qquad t' = Q(t) \qquad u' = R(x,t,u) \tag{4}$$

which relates (1) and the equation

$$\frac{\partial u'}{\partial t'} = x'^{1-M_2}\frac{\partial}{\partial x'}\left[x'^{N_2-1}f(u')\frac{\partial u'}{\partial x'}\right]. \tag{5}$$

We point out that, in general, the functions P and Q also depend on x, t and u, but because the right-hand side of (1) is a polynomial in the derivatives of u with respect to x, it can be shown [10], that $P_u = Q_x = Q_u = 0$. We also point out that by non-degenerate point transformations we mean

$$\frac{\partial(P,Q,R)}{\partial(x,t,u)} \neq 0 \qquad \text{and} \qquad \frac{\partial(P(x,t),Q(t))}{\partial(x,t)} \neq 0.$$

To ensure that the above conditions hold we must have

$$P_x \neq 0 \qquad Q_t \neq 0 \qquad R_u \neq 0.$$

The procedure for determining transformations of the class given by (4) is well explained in [11]. Using transformations (4) we can derive the corresponding transformations for $u'_{t'}$, $u'_{x'}$ and $u'_{x'x'}$. Then substituting in (5) and using (1) we obtain an identity of the form

$$E(x,t,u,u_x,u_{xx}) = 0. \tag{6}$$

Equating the coefficients of u_x^2 and u_{xx} in identity (6) we obtain

$$P = P(x) \qquad Q = t \qquad R = A(x)u + B(x) \tag{7}$$

$$f(u') = x^{N_1-M_1}P^{M_2-N_2}P_x^2 f(u) \tag{8}$$

where A and B are functions to be determined. We state that, in general, the function Q is linear in t but a rescaling of t can be usually be replaced by a further rescaling of x. In addition, the coefficient of u_x and the term independent of the derivatives of u give the following two identities:

$$\begin{aligned}&2xA_xPP_xuf' + 2xB_xPP_xf' + (2xA_xPP_x\\ &\qquad -xAPP_{xx} + (N_2-1)xAP_x^2 + (1-N_1)APP_x)f = 0\end{aligned} \tag{9}$$

$$\begin{aligned}&(A_xu+B_x)^2PP_xf' + (AA_{xx}PP_x - AA_xPP_{xx} + (N_2-1)AA_xP_x^2)uf\\ &\qquad +(AB_{xx}PP_x - AB_xPP_{xx} + (N_2-1)AB_xP_x^2)f = 0\end{aligned} \tag{10}$$

where $f' = \mathrm{d}f/\mathrm{d}u$. We employ the identities (8)–(10) to derive the desired point transformations for each of the three exclusive cases. From equation (9) we see that the function $f(u)$ satisfies the differential equation $\lambda_1uf' + \lambda_2f' + \lambda_3f = 0$. From this equation we can deduce the functional form of $f(u)$ [9] which leads us to the cases that are examined in the following analysis.

Case 1. $f(u)$ arbitrary. If $A_x = B_x = 0$ then identity (10) is satisfied and from (9) we get

$$xAPP_{xx} + (1-N_2)xAP_x^2 + (N_1-1)APP_x = 0. \tag{11}$$

Using equations (8) and (11) we can derive the form of the function $P(x)$. We note that the corresponding transformations will hold for an arbitrary function $f(u)$. From equation (8) we have

$$x^{N_1-M_1}P^{M_2-N_2}P_x^2 = C \tag{12}$$

where C is a constant. The following point transformations will hold for any arbitrary function $f(u)$ if $A = 1$, $B = 0$ and $C = 1$. But given a specific function $f(u)$ then there exists a relation between the constants A, B and C which can be found from (8).

Solving equations (11) and (12) leads to the following four transformations which relate (5) and (1):

(TR.1) $x' = \left(\lambda \frac{K_1}{K_2}\right)^{1/K_2} x^{K_1/K_2} \qquad t' = t \qquad u' = Au + B$.

where λ is a constant and $K_i = M_i$ if $M_i \neq 0$ or $K_i = N_i - 2$ if $N_i \neq 2, (i = 1, 2)$, provided that the parameters M_1, M_2, N_1, N_2 satisfy the condition

$$(N_2 - 2)M_1 = (N_1 - 2)M_2 \tag{13}$$

and the constant C which appears in (12) is given by

$$C = \lambda^2 \left(\frac{N_1 - 2}{\lambda(N_2 - 2)}\right)^{\frac{N_2+M_2-2}{N_2-2}} \tag{14}$$

The above transformation is a generalization of the transformation given by (2).

We note that in the case $N_1 = 2$, $M_1 = 0$, $N_2 = 2 - M_2$ or in the case $N_2 = 2$, $M_2 = 0$, $N_1 = 2 - M_1$ we get only one independent equation which enables us to derive the functional form of $P(x)$. This leads to the point transformations (TR.2) and its inverse, (TR.3) which are given below:

(TR.2) $M_1 = 0 \qquad N_1 = 2 \qquad N_2 = 2 - M_2 \qquad M_2 \neq 0$

$$x' = (M_2\sqrt{C}\ln x + \mu)^{1/M_2} \qquad t' = t \qquad u' = Au + B\,.$$

(TR.3) $M_2 = 0 \qquad N_2 = 2 \qquad N_1 = 2 - M_1 \qquad M_1 \neq 0$

$$x' = \mu \exp\left[\frac{\sqrt{C}}{M_1} x^{M_1}\right] \qquad t' = t \qquad u' = Au + B\,.$$

In the case where $M_i = 0$ and $N_i = 2, (i = 1, 2)$ we have the invariant point transformation

(TR.4) $x' = \mu x^{\sqrt{C}} \qquad t' = t \qquad u' = Au + B$

where μ is a constant. In the case where $f(u) = u^n$, we have $A = C^{1/n}$ and $B = 0$ and (TR.4) maps

$$u_t = x[xu^n u_x]_x\,. \tag{15}$$

into itself.

Case 2. $f(u) = u^n$. We substitute $f(u) = u^n$ into identities (8)–(10). The coefficient of u^{n-1} in (9) implies that $B_x = 0$. Hence, $B =$ constant and without loss of generality we can take $B = 0$. Therefore we have $u' = R = A(x)u$ and (8)–(10) take the form

$$A^n = x^{N_1-M_1} P^{M_2-N_2} P_x^2 \tag{16}$$

$$2(n+1)xPP_xA_x - xPP_{xx}A + (N_2 - 1)xP_x^2 A + (1 - N_1)PP_xA = 0 \tag{17}$$

$$PP_xAA_{xx} + nPP_xA_x^2 - PP_{xx}AA_x + (N_2 - 1)P_x^2 AA_x = 0\,. \tag{18}$$

The overdetermined system (16)–(18) enables the desired point transformations to be derived and ultimately imposes restrictions on the functional forms of $P(x)$ and $A(x)$. If $A =$ constant then we recover (TR.1)–(TR.4) in case 1, with $A = C^{1/n}$ and $B = 0$.

Now we examine the case where $A_x \neq 0$. Integration of the system (16)–(18) requires consideration of the cases: (i) $n = -1$ and (ii) $n \neq -1$. Moreover, conditions on the parameters M_1, N_1, M_2, N_2 lead to different subcases.

If $n=-1$, equations (16)–(18) lead us to three different point transformations which can also be obtained using the results in [8] and the point transformation (2). We have:

(TR.5) $f(u)=u^{-1}$ $\quad N_1=2-M_1 \quad N_2=2 \quad M_1\neq 0 \quad M_2\neq 0$

$$x'=\exp\left[\mp\frac{1}{M_1}x^{M_1}\right] \qquad t'=t \qquad u'=\exp\left[\pm\frac{M_2}{M_1}x^{M_1}\right]u$$

(TR.6) $f(u)=u^{-1}$ $\quad N_1=2 \quad N_2=2-M_2 \quad M_1\neq 0 \quad M_2\neq 0$

$$x'=(\ln x)^{1/M_2} \qquad t'=t \qquad u'=M_2^2x^{M_1}u\,.$$

(TR.7) $f(u)=u^{-1}$ $\quad N_1=N_2=2 \quad M_1,\ M_2$ arbitrary

$$x'=x^{\lambda} \qquad t'=t \qquad u'=\frac{1}{\lambda^2}x^{M_1-\lambda M_2}u\,.$$

We note that (TR.6) is the inverse transformation of (TR.5). In (TR.7), if $M_1=0$ or $M_2=0$, the condition $M_i-N_i+2\neq 0$ which is required to have a transformation from the symmetric nonlinear diffusion equation (3) to the generalized diffusion equation (1) does not hold. Nevertheless, the point transformation (TR.7) is still valid even in the case where $M_1=0$ or $M_2=0$.

If $n\neq -1$, then the system (16)–(18) leads to different functional forms for $P(x)$ and $A(x)$. Consequently, we have the following results:

(TR.8) $f=u^n \quad n=\dfrac{(N_2-2)(2N_1-M_1-4)-M_2(N_1-2)}{M_2(N_1-2)-(N_2-2)(N_1-M_1-2)} \quad N_1\neq 2 \quad N_2\neq 2$

$$x'=(2-N_2)x^{\frac{N_1-2}{2-N_2}} \qquad t'=t \qquad u'=\frac{n+1}{N_1-2}x^{\frac{N_1-2}{n+1}}u\,.$$

From (TR.8) it is interesting to notice some special cases. We see that we have a mapping from $M_1=3N_1-6$ (equation (1)) to $M_2=2-N_2$ (equation (5)) or vice versa if $n=0$, from $M_1=2-N_1$ to $M_2=2-N_2$ if $n=-\frac{4}{3}$ and from $M_1=3N_1-6$ to $M_2=3N_2-6$ if $n=-\frac{4}{5}$. We also note that if $N_1=2+M_1$ then $n=(N_2-M_2-2)/M_2$ (or if $N_2=2+M_2$ then $n=(N_1-M_1-2)/M_1$). If $N_i=2+M_i$ and $M_1=M_2$ then we obtain an invariant transformation with $n=0$. An interesting mapping occurs when $N_2=2+M_2$ and $N_1=M_1=1$. This is the transformation

$$x'=x^{1/M_2} \qquad t'=t \qquad u'=|M_2|xu$$

which relates (5) with $f(u)=u^{-2}$ and $N_2=2+M_2$, with the nonlinear diffusion equation

$$u_t=\left[u^{-2}u_x\right]_x\,. \tag{19}$$

Equation (19) possesses some remarkable properties, see for example in [12]. In particular, it admits an infinite number of Lie–Bäcklund transformations with the employment of a linear recursion operator. Also there exists a 1–1 transformation which maps (19) into the linear diffusion equation $u_t-u_{xx}=0$.

Now if we set $M_1=M_2=0$ in (TR.8) we have $n=-2$. Additionally if $N_1=N_2=N$ then we obtain the point transformation $x'=(2-N)/x,\ t'=t,\ u'=[1/(2-N)]x^{2-N}u$. The latter transformation is a reciprocal in the sense that a double application of it gives the identity transformation. Equation (19) admits this reciprocal transformation ($N=1$). In fact, (TR.8) is a reciprocal transformation even if $M_1\neq 0$ and $M_2\neq 0$ with $N_1=N_2=N$:

(TR.9) $f=u^n \quad n=\dfrac{3N_2-M_2-6}{4+M_2-2N_2} \quad N_1=2-M_1 \quad M_1\neq 0 \quad N_2\neq 2$

$$x' = \left(x^{M_1} + \alpha\right)^{\frac{1}{N_2-2}} \qquad t' = t \qquad u' = \left(\frac{M_1}{N_2-2}\right)^{2/n} \left(x^{M_1} + \alpha\right)^{-\frac{1}{n+1}} u .$$

$$\text{(TR.10)} \quad f = u^n \qquad n = \frac{3N_1 - M_1 - 6}{4 + M_1 - 2N_1} \qquad N_2 = 2 - M_2 \qquad M_2 \neq 0 \qquad N_1 \neq 2$$

$$x' = \left(\frac{M_2}{N_1-2} x^{N_1-2} + \beta\right)^{1/M_2} \qquad t' = t \qquad u' = x^{\frac{N_1-2}{n+1}} u .$$

Successive applications of (TR.9) and (TR.10) lead to the continuous invariant mapping which is presented in [8], provided that the parameters M_i and N_i $(i = 1, 2)$ satisfy condition (13):

$$\text{(TR.11)} \quad f = u^n \qquad n = \frac{3N_1 - M_1 - 6}{4 + M_1 - 2N_1} \qquad N_2 = 2 \qquad M_2 = 0 \qquad N_1 \neq 2$$

$$x' = \exp\left[x^{N_1-2}\right] \qquad t' = t \qquad u' = (N_1 - 2)^{2/n} x^{\frac{N_1-2}{n+1}} u .$$

If $N_1 = 2 + M_1$ and $M_1 \neq 0$, then $n = -2$, and we have a mapping from (15) into (19):

$$\text{(TR.12)} \quad f = u^n \qquad n = \frac{3N_2 - M_2 - 6}{4 + M_2 - 2N_2} \qquad N_1 = 2 \qquad M_1 = 0 \qquad N_2 \neq 2$$

$$x' = (N_2 - 2)^{\frac{2}{M_2 - N_2 + 2}} (\ln x + \beta)^{\frac{1}{N_2-2}} \qquad t' = t \qquad u' = (\ln x + \beta)^{\frac{-1}{n+1}} u .$$

If $N_2 = M_2 + 2$ then $n = -2$ and $|M_2| = 1$. We have $x' = \lambda(\ln x + \beta)$ if $M_2 = 1$ $(N_2 = 3)$ and $x' = \lambda/(\ln x + \beta)$ if $M_2 = -1$, $(N_2 = 1)$. We also note that (TR.12) is the inverse of (TR.11).

Now if $N_1 + M_1 = 2$ or $N_2 + M_2 = 2$ and $n = -\frac{4}{3}$ then the system of differential equations (16)–(18) only has two independent equations. Therefore some of the constants of integration which appear are not forced to be equal to zero in order to satisfy the third equation. In the case where $N_i + M_i = 2$, $(i = 1, 2)$ and $n = -\frac{4}{3}$ the corresponding point transformation can be obtained if we successively apply (TR.9) and (TR.10). In addition, we have the following two transformations for $n = -\frac{4}{3}$:

$$\text{(TR.13)} \quad f(u) = u^{-4/3} \qquad N_2 = 2 \qquad M_2 = 0 \qquad N_1 = 2 - M_1 \qquad M_1 \neq 0$$

$$x' = \exp\left[\frac{M_1}{x^{M_1} + \beta}\right] \qquad t' = t \qquad u' = \left[\frac{x^{M_1} + \beta}{M_1}\right]^3 u .$$

$$\text{(TR.14)} \quad f(u) = u^{-4/3} \qquad N_1 = 2 \qquad M_1 = 0 \qquad N_2 = 2 - M_2 \qquad M_2 \neq 0$$

$$x' = \left[-3M_2(-\tfrac{1}{3}\ln x + c_1)^{-1} + c_2\right]^{\frac{1}{M_2}} \qquad t' = t \qquad u' = \left(-\tfrac{1}{3}\ln x + c_1\right)^3 u .$$

We can see that (TR.14) is the inverse transformation of (TR.13).

It is well known that (3) with $f(u) = u^n$, n arbitrary and $N = 1$ admits a four-parameter group of transformations while in the case where $n = -\frac{4}{3}$, it admits an additional one parameter group [13]. The above point transformations explain why (1) with $f(u) = u^{-4/3}$, $N_1 = 2$, $M_1 = 0$ also admits a five-parameter group of transformations.

Case 3. $f(u) = e^u$. We substitute $f(u) = e^u$ into identities (8)–(10). The coefficient of ue^u in (9) implies that $A_x = 0$. Hence, $A =$ constant and without loss of generality we can take $A = 1$. Therefore we have $u' = R = u + B(x)$ and (8)–(10) take the form

$$e^B = x^{N_1 - M_1} P^{M_2 - N_2} P_x^2 \tag{20}$$

$$2xPP_xB_x - xPP_{xx} + (N_2 - 1)xP_x^2 + (1 - N_1)PP_x = 0 \tag{21}$$

$$PP_xB_{xx} + PP_xB_x^2 - PP_{xx}B_x + (N_2 - 1)P_x^2 B_x = 0 . \tag{22}$$

Similarly as in the previous case, we classify the functional forms of $P(x)$ and $B(x)$ which satisfy the overdetermined system (20)–(22). If B is a constant then we recover the transformations (TR.1)–(TR.4) in case 1, with $A = 1$ and $B = \ln C$.

In the case where $B_x \neq 0$, integration of the system (20)–(22) depends on different conditions on the parameters $M_1,\ M_2,\ N_1,\ N_2$. We can therefore deduce the following transformations:

(TR.15) $f(u) = e^u \qquad N_1 \neq 2 \qquad N_2 \neq 2$

$$x' = x^{\frac{2-N_1}{N_2-2}} \qquad t' = t \qquad u' = u + 2\ln\left|\left(\frac{N_1-2}{N_2-2}\right)x^{N_1-2}\right|$$

provided that the parameters $M_1,\ M_2,\ N_1,\ N_2$ satisfy the condition

$$(N_1-2)(N_2-2) = M_1(N_2-2) + M_2(N_1-2) \tag{23}$$

(TR.16) $f(u) = e^u \qquad N_2 = 2 - M_2 \qquad N_1 = 2 + \frac{1}{2}M_1 \qquad M_1 \neq 0 \qquad M_2 \neq 0$

$$x' = \left(2\frac{M_2}{M_1}x^{M_1/2} + \mu\right)^{1/M_2} \qquad t' = t \qquad u' = u + \tfrac{1}{2}\ln x^{M_1}.$$

(TR.17) $f(u) = e^u \qquad N_1 = 2 + 2M_1 \qquad N_2 = 2 + 2M_2 \qquad M_1 \neq 0 \qquad M_2 \neq 0$

$$x' = \left(\frac{M_1}{M_2}\right)^{2/M_2} x^{-M_1/M_2} \qquad t' = t \qquad u' = u + 2\ln x^{M_1}$$

(TR.18) $f(u) = e^u \qquad N_1 = 2 + \frac{1}{2}M_1 \qquad M_1 \neq 0 \qquad M_2 = 0 \qquad N_2 = 2$

$$x' = \exp\left[\left|\frac{2}{M_1}\right| x^{M_1/2}\right] \qquad t' = t \qquad u' = u + \tfrac{1}{2}\ln x^{M_1}.$$

(TR.19) $f(u) = e^u \qquad N_1 = 2 \qquad M_1 = 0 \qquad N_2 = 2 + \frac{1}{2}M_2 \qquad M_2 \neq 0$

$$x' = (\ln x)^{2/M_2} \qquad t' = t \qquad u' = u - \ln\left[\tfrac{1}{4}M_2^2 \ln x\right].$$

We note that if we set $N_1 = N_2$ in (TR.15) and $M_1 = M_2$ in (TR.17) then both become reciprocal transformations. We also note that (TR.19) is the inverse of (TR.18).

3 Applications of point transformations

Many exact solutions to (1) with $f(u) = u^n$ have been found. A number of similarity solutions are discussed in [4–7]. The point transformations which are derived here may be used to obtain new solutions from these known solutions. We present three simple examples using some of the solutions which appear in [4].

(i) If $f(u) = u^n,\ n \neq 0$, then (1) has the solution

$$u = t^{-\frac{M_1}{(n+1)M_1-N_1+2}}\left[c + \frac{n}{(n+1)(M_1-N_1+2)^2}x^{M_1-N_1+2}t^{-\frac{M_1-N_1+2}{(n+1)M_1-N_1+2}}\right]^{1/n}$$

$$N_1 \neq M_1 + 2 \tag{24}$$

$$u = t^{-1/n}\left[c - \frac{1}{M_1}(\ln x)t^{-\frac{1}{nM_1}}\right]^{1/n} \qquad N_1 = M_1 + 2 \qquad M_1 \neq 0. \tag{25}$$

If we set $N_1 = 2 - M_1$ and $n = -\frac{4}{3}$ and use solution (24) and the transformation (TR.13) we obtain the solution

$$u = [\sqrt{t}\ln x]^{-3}\left[c + \frac{t^{-3}}{M_1^2}\left(\frac{M_1}{\ln x} - \beta\right)^2\right]^{-3/4} \tag{26}$$

of (15) with $n = -\frac{4}{3}$. Setting $n = -2$, solution (25) and transformation (TR.8) with $N_1 = 2 + M_1$ and $N_2 = M_2 = 1$ lead us to the solution

$$u = \left|\frac{1}{M_1}\right| \frac{1}{x} \sqrt{\frac{t}{c - \frac{\sqrt{t}}{M_1^2} \ln x}}. \tag{27}$$

(ii) If $f(u) = u^n$, $n = (N_1 - M_1 - 2)/M_1$ equation (1) has the exact solution

$$u = \mathrm{e}^{-kt} \left(\frac{k}{M_1^2} x^{M_1 - N_1 + 2} \mathrm{e}^{nkt} + c \right)^{1/n}. \tag{28}$$

Setting $N_1 = 2 - M_1$ ($n = -2$) and using solution (28) and point transformation (TR.9) with $N_2 = 2 + M_2$ we derive the solution

$$u = \frac{M_2 x^{M_2}}{\sqrt{c M_1^2 \mathrm{e}^{2kt} + k(x^{M_2} - \alpha)^2}} \tag{29}$$

of the equation

$$u_t = x^{1-M_2} [x^{1+M_2} u^{-2} u_x]_x . \tag{30}$$

(iii) If $f(u) = u^n$, $n < 0$ and $M_1 = 0$, $N_1 \neq 2$ equation (1) has the similarity solution

$$u = \left[\frac{-n x^{2-N_1}}{t(N_1 - 2)^2} \right]^{1/n}. \tag{31}$$

Setting $n = -\frac{3}{2}$ and using solution (31) and (TR.11) we obtain the solution

$$u = \left[\frac{2t}{3(\ln x)^2} \right]^{2/3} \tag{32}$$

of (15).

We point out that in example (i) solution (26) cannot be obtained directly from solution (24). Similarly in example (ii) solution (29) cannot be obtained directly from solution (28) and in example (iii) solution (32) cannot be obtained directly from solution (31).

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